Lectures on the three–dimensional non–commutative spheres

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Abstract

These are expanded notes for a short course given at the Universidad Nacional de La Plata. They aim at giving a self-contained account of the results of Alain Connes and Michel Dubois–Violette.

1 Introduction

The description of the moduli space of non–commutative three–dimensional spheres is a beautiful piece of mathematics, first presented by Alain Connes and Michel Dubois–Violette in [1, 2, 3]. In non–commutative geometry, it is the first exhaustive study of a category of non–commutative spaces.

Non–commutative spaces have been studied both from the mathematical and physical perspectives. In the mathematical realm, a strong motivation has been the study of "bad" quotients of spaces, like foliations with dense leaves and quotients by ergodic group actions.

Physical motivations have come from the realm of quantum gravity, since absolute limitations on the measurements of positions of events might translate in non-commuting coordinates. In string theories, background gauge field may induce non–commutation of position fields.

The most complete presentation of non-commutative geometry can be found in the book of the creator of the field, Alain Connes [4], but the shorter presentation of [5] provides an interesting introduction. Finally, a panorama of the domain and its applications can be found in [6].

I will first briefly motivate the definition of non-commutative spheres, showing in particular that ordinary spheres as well as some non-commutative deformations are included. In one and two dimensions, the only objects which satisfy the conditions are the commutative spheres, so that we have to go to three dimensions to have non-trivial solutions. Then the moduli space of three-dimensional non-commutative spheres is introduced, the corresponding

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quadratic algebras are described. A large place is devoted to the geometric data, which allows to show that differing parameters really describe differing algebras, with some lights about the special cases. The last sections introduce the covering of the generic quantum spheres by a cross-product algebra of functions with $\mathbb{Z}$.

These lecture notes use two principles to reach a fairly complete treatment in a reasonable space. Avoid any use of the transcendental functions and only present one of the different possible presentations. In particular, I always work with the variables which are used only for the complexified moduli space in [3]. This means that the generator of the algebra are not self-adjoint and it requires some care to express the reality conditions, but it greatly simplifies other aspects.

2 Basic facts in non–commutative geometry

In ordinary differential geometry, a large use is made of the differential forms which allow to define the De Rham complex. Forms can be considered as elements of an algebra generated by the functions and symbols df for any function f with the relations

$$d1 = 0, \quad d(fg) = (df)g + f(dg), \quad \text{(Leibniz rule)}.$$

The differential d can be extended to these forms by making it obey a graded Leibniz rule and imposing that $d^2 = 0$. The classical De Rham complex is obtained adding the commutation rules

$$f(dg) = (dg)f, \quad df\ dg = -dg\ df$$

However these rules are inconsistent if the product of f and g is not commutative. An important step has therefore been to realize that cyclic symmetry is sufficient to establish many properties, giving rise to cyclic (co-)homology.

A fundamental object for any non–commutative space $\mathcal{A}$ is the K-homology, which takes different forms in even and odd dimensions. The even dimensional case is linked to the classification of finitely generated projective modules. Projective modules over the algebra of functions can be realized as the vector space of sections of vector bundles and therefore allow to define direct analogs of K-groups. The trivial modules corresponding to trivial bundles are simply powers $\mathcal{A}^n$. The projective modules are defined by projection operators e in $M_n(\mathcal{A})$ singling out a submodule of $\mathcal{A}^n$. Instead of the projector e, which satisfy $e^* = e$ and $e^2 = e$, it can be easier to deal with the involution $s = 2e - 1$ which is also hermitian and satisfy $s^2 = 1$.

In the odd dimensional case, the generators of K-theory are unitary operators U in $M_n(\mathcal{A})$. They are characterized by $UU^* = U^*U = 1$. In both case there is a coupling between the K-groups and the cyclic homology, the Chern characters. To describe them, it is convenient to introduce the operation $\odot$ in $M_n(\mathcal{A}) \simeq \mathcal{A} \otimes M_n(\mathbb{C})$:

$$\begin{aligned} \odot : \quad & M_n(\mathcal{A}) \times M_n(\mathcal{A}) \to M_n(\mathcal{A} \otimes \mathcal{A}) \\ & (a \otimes m, b \otimes n) \to (a \otimes m) \odot (b \otimes n) = (a \otimes b) \otimes (m \circ n) \end{aligned} \tag{1}$$

With a trace acting only on the matrix part of the expressions, we have the following formulas for the Chern characters:

$$\mathrm{Ch}_{2n}(s) = \mathrm{Tr}(\overbrace{s \odot s \odot \cdots \odot s}^{2n+1}) \tag{2}$$

$$\begin{aligned} \mathrm{Ch}_{2n+1}(U, U^*) &= \mathrm{Tr}(\overbrace{\widehat{U \odot U^*} \odot \cdots \odot \widehat{U \odot U^*}}^{n+1} \\ &\quad - \overbrace{\widehat{U^* \odot U} \odot \cdots \odot \widehat{U^* \odot U}}^{n+1}) \end{aligned} \tag{3}$$

3 Definition of non–commutative spheres.

The sphere of dimension n has the property that the cohomology and the homology is supported only in the dimensions 0 and n. Equipped with the notion of Chern character in cyclic cohomology, the following definition has been proposed [1] for the non–commutative spheres.

Definition 1 *An even non-commutative sphere of dimension* $2n$ *is an algebra generated by the matrix elements of a hermitian involution* s *such that* $\mathrm{Ch}_{2p}(s) = 0$ *for all* p *smaller than* n *and* $\mathrm{Ch}_{2n}(s)$ *is the volume form of the sphere.*

An odd non-commutative sphere of dimension $2n+1$ *is an algebra generated by the matrix elements of a unitary* U *together with it hermitian conjugate* U^* *such that* $\mathrm{Ch}_{2p+1}(U, U^*) = 0$ *for all* p *smaller than* n *and* $\mathrm{Ch}_{2n+1}(U, U^*)$ *is the volume form of the sphere.*

In many cases, it will be convenient to separate the conditions that U is unitary or s is an involution in a first one that s^2 or $UU^* = U^*U$ be proportional to the identity

$$s^2 = C\mathbb{I}, UU^* = U^*U = C\mathbb{I} \tag{4}$$

and a second one that this proportionality constant is 1. The advantage is that the first part is purely quadratic in the matrix elements of s or U and U^*. The proportionality constant C can be seen to be central in the algebra by considering the products s^3, UU^*U or U^*UU^*. It is therefore always possible to consider the quantum sphere as a quotient of a quantum affine space which is a quadratic algebra by the ideal generated by $C - 1$, where C is a central element quadratic in the generators.

With this formula, it is easy to see that a quantum sphere allows to define a quantum sphere with one dimension more, by an operation called suspension. If we add a central hermitian element x to the algebra, $U = x\mathbb{I} + is$ and $U^* = x\mathbb{I} - is$ satisfy $UU^* = U^*U = (x^2 + C)\mathbb{I}$, with the new constant $C' = x^2 + C$. The odd Chern forms are cup products of the even Chern forms with the homology of $\mathbb{C}[x]$, so that if s defines a sphere of dimension $2n$, U and U^* define a sphere of dimension $2n+1$. If we start from an odd sphere, we have to double the matrix dimension to define s through:

$$s = \begin{pmatrix} x\mathbb{I} & U \\ U^* & -x\mathbb{I} \end{pmatrix} \tag{5}$$

The quadratic relation is satisfied again with $C' = C + x^2$, $\operatorname{Tr} s$ is evidently 0 and the Chern forms of higher rank are again cup products of the odd Chern forms for U with the ones of the line.

In [1], it was shown that the commutative spheres, as well as some non–commutative deformations, satisfy this definition. Indeed, the matrix s or U can be defined in terms of the Clifford algebra of the space of dimension $2n+1$:

$$s = \sum_{j=0}^{2n} x_j \gamma_j, \tag{6}$$

$$U = x_0 \mathbb{I} + i \sum_{i=1}^{2n+1} x_j \gamma_j. \tag{7}$$

The properties of the Clifford algebra ensure that

$$s^2 = \sum_{j=0}^{2n} x_j^2 \mathbb{I}, \tag{8}$$

$$UU^* = U^* U = \sum_{j=0}^{2n+1} x_j^2 \mathbb{I}, \tag{9}$$

so that the defining properties of the involutions or the unitary operators are satisfied on the sphere. The properties of the Clifford algebra allow to show that the Chern characters of low order are. In the even case, the Chern character involves the trace of an odd number of γ matrices. This is zero unless the number of matrices is at least equal to $2n+1$, in which case it is proportional to the totally antisymmetric symbol. The low order Chern characters are therefore zero and we get for the character Ch_{2n} that it is proportional to the volume form. In the odd case, it can be seen that the two terms annihilate themselves if there are not an odd number of terms proportional to x_0. Again we remain with the trace of an odd number of γ matrices, and the lowest order non-zero Chern character is Ch_{2n+1} with a single term proportional to x_0. Again it can be shown that this non-zero Chern character is proportional to the volume form. One could also obtain the commutative spheres by iterated suspension of the circle, the sphere of dimension 1.

In low dimensions, the only possible spheres are the commutative ones. In dimension one, there are no conditions and we have the algebra generated by a single unitary element u which is the algebra of function on the circle. In dimension two, the generator s must be taken as a two by two matrix and it satisfies the condition $\mathrm{Ch}_0(s) = 0$ which is simply that it has zero trace. We can therefore write it, allowing for the hermitian constraint:

$$s = \begin{pmatrix} x & y \\ y^* & -x \end{pmatrix}. \tag{10}$$

The condition $s^2 = \mathbb{I}$ then implies that x, y and y^* mutually commute and satisfy $x^2 + yy^* = 1$. The first non-trivial case is the three–dimensional one.

4 Moduli space of non-commutative three–spheres

For three-dimensional spheres, the generator of K–theory is a unitary operator U in $M_2(\mathcal{A})$. In terms of the Pauli matrices σ, U and its adjoint are written:

$$U = z_0\mathbb{I} + i\sum_{j=1}^{3} z_j\sigma_j. \tag{11}$$

$$U^* = z_0^*\mathbb{I} - i\sum_{j=1}^{3} z_j^*\sigma_j. \tag{12}$$

The defining equations of the three–sphere are invariant by left and right multiplication of the matrix U by unitary matrices A and B. This corresponds to an arbitrary $SO(4)$ rotation mixing the algebra elements z_μ and their multiplication by a common phase.

With these parameters, the Chern character $\mathrm{Ch}_1(U, U^*)$ reads:

$$\sum_{\mu=0}^{3} z_\mu \otimes z_\mu^* - \sum_{\mu=0}^{3} z_\mu^* \otimes z_\mu. \tag{13}$$

For this Chern character to be zero, the z_μ^* must be linear combinations of the z_μ. If the z_μ are linearly independent, there exists a unique matrix Λ such that

$$z_\mu^* = \Lambda_{\mu\nu} z_\nu \tag{14}$$

The case where the z_μ are linearly dependent is not really important since it does not allow to form spaces with non–trivial three–volumes. We can use the redefinition by a four dimensional rotation to put some generators to zero and remain with independent generators. A formula similar to eq. (14) can be written for the non-zero generators.

Inserting in eq. (13) shows that Λ is symmetric and taking the adjoint of eq. (14) shows that it is unitary. The real and imaginary part of Λ are real symmetric matrices, the unitarity of Λ implies that they commute and they can therefore be simultaneously diagonalized by a four dimensional rotation:

$$z_\mu^* = \lambda_\mu z_\mu, \tag{15}$$

without summation over μ. The four eigenvalues of Λ are unit norm complex numbers and they can all be multiplied by a common phase by a redefinition of U. The moduli space is therefore of dimension three.

After the diagonalization, we keep the freedom of changing the order of the eigenvalues. There is a natural cyclic order on the circle, but the determination of the starting point, which can be set to one, remains. This can be settled if we impose that the imaginary parts of λ_2/λ_0 and λ_3/λ_1 are positive. In terms of the parameters ϕ_i such that:

$$\lambda_0 = 1, \quad \lambda_i = e^{i\phi_i}, \tag{16}$$

this gives two possible set of constraints:

$$0 \leq \phi_1 \leq \phi_2 \leq \phi_3 \leq \pi, \tag{17}$$

$$\phi_1 \leq \phi_2 \leq \pi \leq \phi_3 \leq \pi + \phi_1. \tag{18}$$

The boundaries of this fundamental domain correspond to either the equality of two λ or to a relation $\lambda_i = -\lambda_j$.

It remains to show that different parameters really correspond to distinct algebras and that an algebra can be realized with any choice of these parameters.

5 The quadratic algebras.

The unitarity condition on U can be expanded on the basis of the matrices $\mathbb{I}$ and σ_i. The $\mathbb{I}$ term is identical for the UU^* and U^*U products and gives the quadratic relation:

$$\sum_{\mu=0}^{3} \lambda_\mu z_\mu^2 = 1. \tag{19}$$

This is the inhomogeneous relation in the algebra and its left hand side is automatically central.

There remains six equations which can be deduced from the following two by a circular permutation of the indices 1, 2 and 3.

$$\begin{aligned} \lambda_0 z_1 z_0 - \lambda_1 z_0 z_1 + \lambda_3 z_2 z_3 - \lambda_2 z_3 z_2 &= 0, \\ -\lambda_1 z_1 z_0 + \lambda_0 z_0 z_1 + \lambda_2 z_2 z_3 - \lambda_3 z_3 z_2 &= 0. \end{aligned} \tag{20}$$

These two equations are related if we take new generators $z'_0 = -\lambda_0 z_0$, $z'_i = \lambda_i z_i$, and define new parameters $\lambda'_\mu = \lambda_\mu^{-1}$. This is linked to the symmetry of the roles of U and U^* in the definition of the algebra. The primed parameter determine U^* in the same manner that the other ones determine U. This only changes the sign of the Chern character, that is the orientation of the sphere. We have therefore an isomorphism between the algebra of a sphere and the algebra for the sphere with all of the parameters λ replaced by their inverses. Since the λ are unit norm complex numbers, taking the inverses or the complex conjugates are equivalent.

The sum and the difference of the equations (20) express them in terms of commutators and anticommutators:

$$\begin{aligned} (\lambda_1 - \lambda_0)[z_0, z_1]_+ &= (\lambda_2 + \lambda_3)[z_2, z_3]_-, \\ (\lambda_1 + \lambda_0)[z_0, z_1]_- &= (\lambda_3 - \lambda_2)[z_2, z_3]_+. \end{aligned} \tag{21}$$

In this form, it is clear that when the λ's are equal, all these relations reduce to the mutual commutation of all the generators z_μ, so that we obtain the commutative sphere.

We can also remark that the algebra and its opposite defined by $a^\circ b^\circ = (ba)^\circ$ are isomorphic. Indeed, the commutators change sign and the anticommutators are unchanged in the change to the opposite algebra and this can be compensated in eqs. (21) by changing the sign of any of the generators, since each of them appears once in each of the equations (21). Changing the sign of two generators yields an automorphism.

In the form (21), changing the scale of the generators can allow to put one of the equations to a standard form, if we are not in one of the special cases where some of the sums or differences of the λ are zero. According to the one put to standard form, we can obtain a Sklyanin algebra or a similar one given by the equations:

$$\begin{aligned} [Z_0, Z_1]_- &= [Z_2, Z_3]_+, \\ (\lambda_1 - \lambda_0)(\lambda_2 - \lambda_3)[Z_0, Z_1]_+ &= (\lambda_1 + \lambda_0)(\lambda_3 + \lambda_2)[Z_2, Z_3]_-. \end{aligned} \tag{22}$$

Again, the equations with circular permutations of the indices are understood. From now on, we will adopt the convention that the indices k, l, m will stand for $1, 2, 3$ or any of its circular permutations. In any expression with these three indices, a sum over k or a product over k will stand for the sum or the product on the three possible permutations. If we introduce the quantities $a_k = (\lambda_k + \lambda_0)(\lambda_l + \lambda_m)$, the second set of equations takes the form

$$(a_m - a_l)[Z_0, Z_k]_+ = a_k[Z_l, Z_m]_-. \tag{23}$$

This shows that the quadratic part of the algebra only depends on two parameters, since these equations are invariant by a rescaling of the a_i. This corresponds to the fact that there is always a second central element quadratic in the generators, so that the sphere can be deformed by modifying the inhomogeneous condition by adding this other central element. This second central element gives a foliation of the sphere in tori.

The explicit relation between z_μ and Z_μ is given by

$$Z_\mu = \rho_\mu z_\mu, \tag{24}$$

with ρ satisfying the following equations:

$$\rho_0\rho_k(\lambda_m - \lambda_l) = \rho_m\rho_l(\lambda_0 + \lambda_k) \tag{25}$$

We can choose

$$\begin{aligned} \rho_0^2 &= \prod(\lambda_0 + \lambda_k), \quad \rho_k^2 = (\lambda_0 + \lambda_k)(\lambda_l - \lambda_k)(\lambda_k - \lambda_m) \\ \rho_0\rho_1\rho_2\rho_3 &= \prod_k[(\lambda_0 + \lambda_k)(\lambda_m - \lambda_l)] \end{aligned} \tag{26}$$

It is interesting to note that with this choice for ρ_μ and the normalization $\prod \lambda_\mu = 1$, the new generator Z_μ are hermitian or anti-hermitian.

$$Z_\mu^* = \rho_\mu^* z_\mu^* = \frac{\lambda_\mu \rho_\mu^*}{\rho_\mu} Z_\mu \tag{27}$$

The relation does not depend on the choice of square roots in the definition of ρ_μ. One has for the unit norm λ:

$$\frac{\lambda_\mu + \lambda_\nu}{\lambda_\mu^* + \lambda_\nu^*} = \frac{\lambda_\mu + \lambda_\nu}{1/\lambda_\mu + 1/\lambda_\nu} = \lambda_\mu\lambda_\nu \tag{28}$$

The square of the factor between Z_μ and Z_μ^* is one. The product of these factors however is -1 using the constraint on the product of the ρ. This result could be anticipated by considering the adjoint of eqs. (22). In the case of eq. (17), one can further show that all the new generators are hermitian except Z_2.

With this presentation of the algebra, the center of the algebra contains three elements Q_k, whose sum is zero:

$$Q_k = (a_m - a_l)(Z_0^2 + Z_k^2) + a_k(Z_m^2 - Z_l^2) \tag{29}$$

The proof uses the identities:

$$[Z_\mu, Z_\nu^2]_- = [[Z_\mu, Z_\nu]_-, Z_\nu]_+ = [[Z_\mu, Z_\nu]_+, Z_\nu]_-. \tag{30}$$

By choosing the appropriate one of these two possibilities for each term of the commutator of Z_μ and Q_k, the identities (22,23) allow to obtain a_k or $a_l - a_m$ as common factor of double (anti-)commutators which sum up to zero.

6 Geometric data.

In order to show that the algebras associated to distinct parameters a_i are really inequivalent, we introduce the geometric data. The quadratic relations R defining the algebra are elements of the tensor product $V \otimes V$ of the vector space of the generators of the algebra. They define quadratic equations in the product of projective spaces $PV^* \times PV^*$. The resulting space Γ define a correspondence σ between the projection of Γ on the first and the second factor. The association of the projection E of Γ and the correspondence σ defined by Γ is the geometric data. Isomorphic algebras will give isomorphic geometric data, so that we can show that some quadratic algebras are inequivalent.

These geometric data, plus the line bundle stemming from the canonical one on PV^*, will furthermore allow to represent the algebra on a space of sections of line bundles.

It appears that the first form of the relations (20) is the most interesting for the determination of the geometric data. If we denote by (y_μ) and (y'_μ) coordinates in the first and second PV^* factor in a basis dual to the z_μ, the equations of Γ read:

$$\begin{aligned} \lambda_0 y_1 y'_0 - \lambda_1 y_0 y'_1 + \lambda_3 y_2 y'_3 - \lambda_2 y_3 y'_2 &= 0, \\ -\lambda_1 y_1 y'_0 + \lambda_0 y_0 y'_1 + \lambda_2 y_2 y'_3 - \lambda_3 y_3 y'_2 &= 0, \end{aligned} \tag{31}$$

plus the one deduced by circular permutation of the indices (1,2,3). The point of E are such that there exist a non zero solution in (y'_0, y'_1, y'_2, y'_3). This means that the following matrix is of rank less than four:

$$\begin{pmatrix} \lambda_0 y_1 & -\lambda_1 y_0 & -\lambda_2 y_3 & \lambda_3 y_2 \\ \lambda_0 y_2 & \lambda_1 y_3 & -\lambda_2 y_0 & -\lambda_3 y_1 \\ \lambda_0 y_3 & -\lambda_1 y_2 & \lambda_2 y_1 & -\lambda_3 y_0 \\ -\lambda_1 y_1 & \lambda_0 y_0 & -\lambda_3 y_3 & \lambda_2 y_2 \\ -\lambda_2 y_2 & \lambda_3 y_3 & \lambda_0 y_0 & -\lambda_1 y_1 \\ -\lambda_3 y_3 & -\lambda_2 y_2 & \lambda_1 y_1 & \lambda_0 y_0 \end{pmatrix} \tag{32}$$

Each of the fifteen four by four determinants that can be extracted from this matrix must be simultaneously zero. This calculation is made manageable by the factorization of the four three by three minors extracted from the first three lines of the matrix: they all have the common quadratic factor $y_0^2+y_1^2+y_2^2+y_3^2$. Similarly, the three by three minors extracted from the last three lines have the common factor $\sum \lambda_\mu^2 y_\mu^2$. Therefore on the curve defined by the intersection of these two quadrics, six of the four by four submatrices have determinant 0. Furthermore, since the upper and the lower subblock are of rank 2, it will not be necessary to verify that the remaining nine submatrices are of determinant 0. Indeed, apart from isolated points on the curve, the third line of each group is a linear combination of the two others. If the determinant of the matrix formed with the lines 1,2,4,5 is 0, we therefore have that they generate a space of dimension at most 3 and all other determinants are also zero. Continuity allows to ensure that these other determinants remain zero on the points where the lines 1 and 2 for example are proportional. That the determinant of the matrix formed with the lines 1,2,4,5 is zero can be seen by writing it:

$$(\lambda_0\lambda_3+\lambda_1\lambda_2)\left[(y_1^2+y_2^2)(\lambda_0^2y_0^2+\lambda_3^2y_3^2)-(y_0^2+y_3^2)(\lambda_1^2y_1^2+\lambda_2^2y_2^2)\right] \tag{33}$$

The two quadratic equations define a curve which is generically an elliptic one:

$$\begin{aligned} y_0^2+y_1^2+y_2^2+y_3^2 &= 0, && (34)\\ \lambda_0^2y_0^2+\lambda_1^2y_1^2+\lambda_2^2y_2^2+\lambda_3^2y_3^2 &= 0. && (35) \end{aligned}$$

If different λ^2 become equal, this curve decompose in a product of rational curves.

Suppose that equation (35) is not verified. We then have three necessary conditions stemming from the determinants including the lines 4,5,6.

$$\begin{aligned} (\lambda_0^2-\lambda_1^2)y_0y_1 &= (\lambda_2^2-\lambda_3^2)y_2y_3\\ (\lambda_0^2-\lambda_2^2)y_0y_2 &= (\lambda_3^2-\lambda_1^2)y_3y_1\\ (\lambda_0^2-\lambda_3^2)y_0y_3 &= (\lambda_1^2-\lambda_2^2)y_1y_2 && (36) \end{aligned}$$

Again, these equations only depend on the squares of the λ_μ. In the case where all the squares are distinct, these homogeneous equations have exactly eight projective solutions. The four trivial solutions P_μ such that only y_μ is non zero and four other ones which can be deduced from one of them by changing the sign of a pair of y_μ. However these additional solutions belong to the generic variety and we do not have to consider them independently. Indeed, in the case where all the y_μ are not zero, the product of the first two equations for example gives the ratio y_0^2/y_3^2 so that we have:

$$\begin{aligned} y_0^2 &= (\lambda_2^2-\lambda_3^2)(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2^2)\\ y_1^2 &= (\lambda_2^2-\lambda_3^2)(\lambda_0^2-\lambda_2^2)(\lambda_0^2-\lambda_3^2)\\ y_2^2 &= (\lambda_0^2-\lambda_1^2)(\lambda_3^2-\lambda_1^2)(\lambda_0^2-\lambda_3^2)\\ y_3^2 &= (\lambda_0^2-\lambda_1^2)(\lambda_0^2-\lambda_2^2)(\lambda_1^2-\lambda_2^2) && (37) \end{aligned}$$

The expressions $\sum y_\mu^2$ and $\sum \lambda_\mu^2 y_\mu^2$ are polynomials of degree two in say λ_0^2 which vanish for the three values λ_i^2, therefore they are zero. The only particular cases to consider are therefore the four projective points with only one non-zero coordinate. It is easy to verify that each of these points is indeed in the variety E and that they are associated to themselves by the correspondence σ.

The case where the equation (34) is not verified is completely similar. In fact, a change of variable allows to exchange the upper and lower block of the matrix, so that no new calculations are necessary to ensure that in the generic case, the variety E is the union of the elliptic curve defined by the equations (34,35) and the four points P_μ. The correspondence σ is the identity on the points P_μ. The geometric data for the opposite algebra involve the same variety E but the inverse correspondence σ^{-1}. The isomorphism of the algebras of the three-spheres with their opposites implies that σ^{-1} can be obtained from σ simply by conjugating with the change of sign of y_0. This identifies σ as a translation of the elliptic curve. We therefore have two parameters to describe the geometric data, the modulus of the elliptic curve and the parameter of the translation.

When different λ_μ^2 become equal, the situation is very similar. The equations (34,35) define a curve which now split in different rational components. As in the generic case, if one of these equations is not satisfied, the equations (36) or a similar set are necessarily satisfied. The solutions are either included in the generic variety or have some zero components. Substituting in the matrix allows to show that these points with some zero components really are in the characteristic variety and that the correspondence is either the identity or a symmetry with some of the coordinates changing sign. The full description of these non-generic cases is better done after the study of symmetries which allow to relate some cases.

7 Reality and symmetries.

Before completing the analysis of the geometric data, we have to consider the reality conditions and the symmetries for the algebras and the relations between different algebras that they allow.

The generators z_μ allow for a simpler description of the characteristic variety, but their conjugation properties make the reality conditions for this variety subtler. The restriction to the space of generators V of the adjunction in the algebra maps to V, so that it defines an antilinear map j. In V^*, the complex conjugation is defined by $j(L)(Z) = \overline{L(j(Z))}$, which gives in coordinates:

$$(y_0, y_1, y_2, y_3) \longrightarrow (\lambda_0^* y_0^*, \lambda_1^* y_1^*, \lambda_2^* y_2^*, \lambda_3^* y_3^*) \tag{38}$$

Since the λ are complex numbers of unit norm, they do not obey the reality condition $x^* = x$ but the one $x^* = x^{-1}$. Combined with the complex conjugation given by eq. (38), this allows to show that the generic variety is indeed invariant by complex conjugation, since the equations (34,35) on the complex conjugates are the complex conjugates of these same equations for the initial point.

From the formula $(ab)^* = b^* a^*$, we obtain that the set of relation R is invariant by the action of $j \otimes j$ followed by the exchange of the factors. This shows that the characteristic variety is invariant by the action of j and that we have:

$$j \circ \sigma = \sigma^{-1} \circ j \tag{39}$$

The algebra automorphisms introduced in section 5 allow to define twisting of the non-commutative spheres which are again non-commutative spheres with related parameters. These automorphisms come from the diagonal transformations in $SO(4)$ and act on the generators by changing the sign of two of them. Now, with any algebra *-automorphism τ one can build a cross-product algebra $\mathcal{A} \times_\tau \mathbb{Z}$ by adjoining to the algebra $\mathcal{A}$ a unitary symbol W satisfying the relation:

$$Wa = \tau(a)W \tag{40}$$

In the cases where the automorphism act by linear transformation on the generators x_i, we can recover an algebra with the same set of generators by considering the subalgebra of the cross-product with generators x_iW. The geometric data for this new algebra can easily be deduced from those of the original algebra. The relations for the new algebra are deduced from the old ones by the action of $\mathbb{I} \otimes \tau^{-1}$ so that the space Γ is changed by the action of the identity on the first factor and the transpose of τ^{-1} on the second. The variety E is unchanged and the correspondence becomes ${}^t\tau^{-1} \circ \sigma$.

In the case of the non-commutative three sphere, τ^2 is the identity so that we can add the condition $W^2 = 1$. We change the matrix generators U to the generator UW. The quadratic relations are deduced from the relation $UU^* = U^*U = \mathbb{I}$ and keep the same form. What will change is the relation between U and U^*.

$$(UW)^* = W^*U^* = WU^* = \tau(U)W. \tag{41}$$

The eigenvalues λ_μ corresponding to the generators on which τ has a non-trivial action will change sign.

In particular, changing the sign of λ_0 and λ_3 and exchanging them allows to relate the algebras corresponding to the case of (18) with those described by (17).

8 Special cases.

In the study of the geometric data, these symmetries allow to reduce the number of different case to study. The non-generic cases can be first classified by the number of different values of the λ^2, which can go from 1 to 4. In the cases of two different values of λ^2, the eigenvalues can be split in two groups of two, or one group of one and one of three.

In the case with three different values of λ^2, we can have the two relations $\lambda_1 = \lambda_2$ or $\lambda_1 = -\lambda_2$ but the two can be related by a symmetry. The generic variety stemming from equations (34,35) split in two conics. To the four special points we must add the line through the points P_1 and P_2.

In the case with two different values of λ^2 and unequal groups, again the different possibilities for the sign relation between the λ can be related by the symmetry. The special variety now contain the plane through the three points P_1, P_2 and P_3. The generic variety is now embedded in this plane, where σ is the identity in the simplest case where three λ are equal.

The case of say $\lambda_1 = \lambda_2$ and $\lambda_0 = \lambda_3$ divide in three cases. The two cases where either the λ are pairwise equal or opposite can be related by symmetry, but the third possibility is distinct. With $\lambda_0 = -\lambda_3$ and $\lambda_1 = \lambda_2$, two of the equations becomes equal, so that the algebra grows exponentially. On the geometric data, this translates in the apparition of coarse correspondences, where some points map to whole lines.

Finally, when all the λ^2 are equal and can therefore be taken to be 1, there are three cases according to the number of -1. With all 1, we have the commutative case, the entire projective space is the characteristic variety and the correspondence is the identity. The case with two -1 is related to the first one by a symmetry. The case with one -1 is very special. All the a_i are zero and we only have three relations in the algebra. The matrix (32) has only three independent lines and is therefore of maximum rank 3. However, on the surface with equation (34), its rank becomes 2 and the correspondence σ associates with such points whole lines.

9 The fundamental elliptic curve.

In section 5, we saw that the quadratic algebra $\mathbb{R}^4_\lambda$ only depends on two parameters, with all parameters lying on a curve describing a unique quadratic algebra. The identification is linked to a rescaling of the generators such that the algebra is defined by eqs. (22,23), which only depend on three homogeneous parameters. What is most remarkable is that the curve in the space of parameters λ which maps to a given set of the a parameters is identical with the generic part of the characteristic variety. This was first an observation on the identity of the elliptic parameters, but the two curves can be explicitly identified.

The scaled generators of the algebra are linked to a new coordinate system for V^* such that $Y_\mu = \rho_\mu y_\mu$. The equations (34,35) become, after eliminating the denominators:

$$\begin{aligned} \prod_k (\lambda_m - \lambda_l) Y_0^2 + \sum_k (\lambda_0 + \lambda_m)(\lambda_0 + \lambda_l)(\lambda_m - \lambda_l) Y_k^2 &= 0, \\ \lambda_0^2 \prod_k (\lambda_m - \lambda_l) Y_0^2 + \sum_k \lambda_k^2 (\lambda_0 + \lambda_m)(\lambda_0 + \lambda_l)(\lambda_m - \lambda_l) Y_k^2 &= 0. \end{aligned} \tag{42}$$

The important property of these equations is that they are linear combinations of $B_k = Y_0^2 - Y_k^2$. Indeed the sums of the coefficients are polynomials of degree less than 2 in λ_0 which are zero at the three points $-\lambda_k$. The linear combination of the equations (42) such that the coefficient of say Y_3^2 is zero allows to obtain the ratio of B_1 and B_2 on the curve. It turns out to be the same as the one of a_1 and a_2.

The B_k can be converted to the a_k through a linear change of variables with an involutive matrix M:

$$M = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \tag{43}$$

We therefore have identified the two elliptic curves, in the parameter space of the non-commutative spheres and in the characteristic variety.

Another interest of this presentation of the quadratic algebra is that the correspondence σ has a simple form, since the first three relations of the algebra, which are independent on the parameters, are sufficient to determine the image of a point. The image $\sigma(Z)$ of a point is determined by the condition that its image by the matrix $N(Z)$ is zero:

$$N(Z) = \begin{pmatrix} Y_1 & -Y_0 & Y_3 & Y_2 \\ Y_2 & Y_3 & -Y_0 & Y_1 \\ Y_3 & Y_2 & Y_1 & -Y_0 \\ (a_2-a_3)Y_1 & (a_2-a_3)Y_0 & -a_1Y_3 & a_1Y_2 \\ (a_3-a_1)Y_2 & a_2Y_3 & (a_3-a_1)Y_0 & -a_2Y_1 \\ (a_1-a_2)Y_3 & -a_3Y_2 & a_3Y_1 & (a_1-a_2)Y_0 \end{pmatrix}. \tag{44}$$

The first three lines form a system of maximal rank, so that they are sufficient to determine $\sigma(Z)$. Furthermore, if we replace Y_0 by its opposite, the first three lines of $N(Z)M$ read:

$$\begin{pmatrix} \lambda_0 & \lambda_1 & -\lambda_2 & -\lambda_3 \\ \lambda_0 & -\lambda_1 & \lambda_2 & -\lambda_3 \\ \lambda_0 & -\lambda_1 & -\lambda_2 & \lambda_3 \end{pmatrix}. \tag{45}$$

This shows that σ is the product $I \circ I_0$ of two involutions: I_0 changes the sign of Y_0 and I corresponds to taking the inverse of all coordinates in the λ coordinates. These two operations are symmetries for the elliptic curves since they have four fixed points.

On these curves there are three different notions of reality. There is the simple one corresponding to taking real coordinates. The one associated to the λ, which are of unit norm, so that we have $\bar{v} = I(v)$. As I corresponds to taking the opposite in the elliptic curve, they are imaginary points. Finally, the j operation on V^*, linked to the adjunction properties of the algebra, changes the sign of an odd number of the complex conjugated coordinates: the j-real points are again purely imaginary points of the curve, but starting from a different origin. The transformation σ has real, even integer, coefficients, so that it is a translation by a real element of the curve.

10 Uses of the geometric data.

The geometric data allow for the definition of a representation of the algebra. Indeed [7], the algebra maps to a cross-product algebra of sections of line bundles by the map σ. Since the variety E is given as a subvariety of the projective space PV^*, it is equipped with a naturally defined line bundle $\mathcal{L}$, the restriction of the canonical bundle of PV^*. To each element of V, we can associate a section s_V of this bundle $\mathcal{L}$. The generator of the algebra are mapped to products $s_V W$, with the symbol W satisfying the commutation relation $W s_V = s_V \circ \sigma W$. The product of n generators is therefore the product of a holomorphic section of a bundle $\mathcal{L}_n$ by W^n. The bundle $\mathcal{L}_n$ is defined in terms of pullbacks of $\mathcal{L}$ by the maps σ^n:

$$\mathcal{L}_n := \mathcal{L} \otimes \mathcal{L}^{\sigma} \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}} \tag{46}$$

The defining relations of the algebra are satisfied from the definition of σ. However, this cannot provide a faithful representation for the non-commutative spheres, since all central elements of the algebra are represented by 0 in the generic case. Indeed, if a central element would have a non-zero representation, we would get inconsistencies for the divisor of its product with any generator. The trouble can be spotted also from the dimensions of the spaces of products of n factors. The Riemann-Roch theorem shows that, in the generic case, this dimension is at most $4n$ when the dimension of the corresponding space is $\binom{n+3}{3}$ for the quadratic algebra and $(n+1)^2$ for the non-commutative sphere. An other limitation of this representation is that it is not a $*$-representation.

Alain Connes and Michel Dubois-Violette introduced an extension of this construction based on the notion of "central quadratic form". This allows for a representation where the inhomogeneous equation of the algebra (19) is satisfied. The main point is to turn the algebra of the previous representation in a *-algebra. Taking the complex conjugate of holomorphic functions gives antiholomorphic ones. We therefore introduce a second variable to represent the complex conjugate. In a first time, the two variables are better kept independent, so that the space we consider is the product $E \times E$ with the complex conjugation $\tilde{j}$:

$$\tilde{j}(Z, Z') = (j(Z'), j(Z)). \tag{47}$$

In the end, we will restrict to the invariant subspace of $\tilde{j}$, formed of points $(Z, j(Z))$. With c denoting complex conjugation in the image space, the adjunction for the function F is defined by:

$$F^* = c \circ F \circ \tilde{j}. \tag{48}$$

With the following representation for the generators,

$$\rho(Y) = Y(Z)W + W^* Y(Z'), \tag{49}$$

it is easy to see that

$$\rho(Y)^* = \rho(j(Y)). \tag{50}$$

The transformation σ must be extended to $E \times E$. In view of the relation between σ and j (39), we define $\tilde{\sigma}$ so that it commutes with $\tilde{j}$:

$$\tilde{\sigma}(Z, Z') = (\sigma(Z), \sigma^{-1}(Z'). \tag{51}$$

It is natural to suppose that W is a unitary, so that the commutation with W^* gives a translation by $\tilde{\sigma}^{-1}$.

Finally, we want the map (49) to define a representation of the algebra of the non-commutative sphere. Terms $W^*Y_i(Z')Y_j(Z)W$ and $Y_i(Z)WW^*Y_j(Z')$ will correspond to fibers in differing points of E, so that a trivialization is necessary for their comparison. Such a trivialization will also be necessary to allow for the inhomogeneous equation (19).

The product of the line bundle $\mathcal{L}$ on the variable Z and Z' is trivialized by dividing by a quadratic form $Q(Z,Z')$. Then, the quadratic relations read:

$$\begin{aligned}\omega_{ij}\rho(Y_i)\rho(Y_j) &= \omega(Z,\sigma(Z))W^2 + (W^*)^2\omega(\sigma^{-1}(Z'),Z') \\ &\quad + \frac{\omega(Z,Z')}{Q(Z,Z')} + \frac{\omega(\sigma(Z'),\sigma^{-1}(Z))}{Q(\sigma^{-1}(Z),\sigma(Z'))}\end{aligned} \tag{52}$$

The parts proportional to W^2 or $(W^*)^2$ are zero from the defining property of σ. The remaining part is zero if Q is symmetric and satisfies the defining property of a central quadratic form:

$$\omega(Z,Z')Q(\sigma(Z'),\sigma^{-1}(Z)) + \omega(\sigma(Z'),\sigma^{-1}(Z))Q(Z,Z') = 0 \tag{53}$$

In the generic case, this condition can be verified if both Z and Z' belong to the generic curve, for any quadratic form deduced from a central quadratic elements of the algebra. The calculation is done in the coordinates introduced in section 9, for any of the central forms Q_k: the transformation σ has a simple explicit expression and the fact that Z and Z' are on the curve can be expressed simply by substituting $y_0^2 + Ca_k$ to y_k^2. The equation (53) is not verified if Z is a special point and Z' is on the elliptic curve, but this does not preclude the possibility to obtain a representation of the algebra. By specializing eq. (53) to the case $Z' = \sigma(Z)$, we can further prove that $Q(Z,\sigma(Z))$ is zero is σ^4 is not the identity. The calculation of (52) applied to $\omega = \frac{1}{2}Q$ now shows that $\frac{1}{2}Q_{ij}Y_iY_j$ is represented by 1. This representation is a *-representation if Q verifies $Q(\tilde{j}(Z,Z')) = \overline{Q(Z,Z')}$.

If we now specialize to a $\tilde{j}$ invariant subset, i.e., to points of the form $(Z, j(Z))$, we can further demand that Q is strictly positive, so that it defines an hermitian metric on $\mathcal{L}$. This is certainly the case for the central element C of the algebra, which can also be written $\sum z_\mu z_\mu^*$. Division by $Q(Z, j(Z))$ does not introduce any singularity and the algebra of the sphere is faithfully represented in the space of regular sections. We have therefore obtained a representation of the non-commutative spheres for generic parameters. The fact that an elliptic curve is topologically a torus can be used to further understand the structure of the algebra. Essentially, we have a family of non-commutative torus with a parameter determined from the transformation σ.

However, this does not provide for a full representation in every cases. In the case where three of the λ are equal, we have a central element z_0 which play the role of the Planck constant for the $su(2)$ algebra formed by the three others generators. The correspondence σ is the identity, so that the algebra obtained from the geometric data is commutative. This is compatible with the algebra structure because the characteristic variety is the union of a point and the plane $y_0 = 0$: on the plane, the geometric data represent z_0 by 0 and all other generators are zero on the isolated point.

The full structure of this algebra is much more interesting. We get a discrete structure, because for a n dimensional representation of the $su(2)$ algebra, $z_1^2 + z_2^2 + z_3^2 = (n^2-1)z_0^2$. We therefore have a discrete family of fuzzy spheres which converge to the ordinary sphere obtained for $z_0 = 0$.

11 Conclusion

We arrive at the end of this tour of non-commutative three-spheres. The definition of the non-commutative three-spheres has been introduced, the moduli space discovered. A large place has been given to the geometric data, first to provide for intrinsic parameters to single out the quadratic algebras, then to show how they give rise to representations for the algebra of the non-commutative spheres.

In a sequel to these lectures, it would be interesting to evaluate the volume form provided by the Chern form of order 3 and the Jacobian of the transformation to the cross-product algebra $F_u \times_{\sigma,\mathcal{L}} \mathbb{Z}$. In keeping to the spirit of these lectures, we could make a purely algebraic version of the computation of section 12 of [3].

We also would like to build spectral triples for these algebras, in order to complete the description of their geometry. However, one expects that this would be rather difficult. In the cases where the characteristic variety is rational, which have quantum group structures of $SU_q(2)$, the construction has met with some difficulties: in the generic case, we do not have any symmetry to assist in the process.

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